A Review of Geometrical Aspects of Ship Motion in Manoeuvring and Seakeeping, and the Use of a Consistent Notation

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Abstract

The design of ship guidance and motion control systems is generally based on the Newton-Euler equations motion and kinematic transformations. These are naturally formulated in terms of a combination of inertial and body-fixed coordinates—*manoeuvring coordinates*. The data used in this formulation can be obtained from numerical hydrodynamics, experimental hydrodynamics, or both. When numerical hydrodynamics is used, the programs that calculate ship data (added mass, damping, force and motion response operators, *etc.*) use, in general, a formulation of the equations of motion in terms of a different set of coordinates—*seakeeping or perturbation coordinates*. Therefore, appropriate transformations have to be made to formulate consistent models. Details of the derivations of these transformations are often omitted in the literature, and blunders are common.

This report has, therefore, two objectives. The first one is to review the geometrical aspects of ship motion (frames, coordinates and transformations) commonly used in the areas of manoeuvring and seakeeping and introduce the kinematic transformations that relates the coordinates used in these two areas. The second objective is to introduce a notation which is consistent with the coordinates used in both areas.

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1 Introduction

The study of ship dynamics has traditionally been separated into two main areas:

- *Steering and manoeuvrability (Manoeuvring)*,
- *Seakeeping*.

Steering and manoeuvrability—commonly shorten to *manoeuvring*—refers to the study of ship motion in the absence of wave-excitation (calm water). This can be considered in open or confined waters (usually shallow) and at low or high speeds. In these conditions, the motion results from the action of control devices: deflection control surfaces (rudders) and propulsion units. *Seakeeping*, on the other hand, refers to the study of motion, stability and control when there is wave excitation and while the vessel keeps its course and its speed constant (which includes the case of zero speed).

Although both areas are concerned with the same issues (study of motion, stability and control), the separation allows one making different assumptions that simplify and make the study trackable in each case. The foundations of this separation lie in the fact that particular ship operations are commonly performed in particular environmental conditions. A consequence of this separation is the use of different set of coordinates and different reference frames in each of these areas.

In the design of guidance and control systems for ships, it is natural to use manoeuvring coordinates to formulate the Newton-Euler equations of motion (Fossen, 1994; Fossen, 2002). Nowadays, it is a standard practice for ship motion simulation and control to obtain preliminary ship design data from hydrodynamic programs (Smogeli *et al.*, 2005; Fossen, 2005). Most hydrodynamic programs, however, use seakeeping coordinates; and therefore, it is necessary to transform the data to obtain consistent models. This issue is the main motivation for this report because details of the derivations of these transformations are often omitted in the literature, and blunders are common.

The report has two objectives. The first one is to review the geometrical aspects of ship motion (frames, coordinates and transformations) commonly used in the areas of manoeuvring and seakeeping and derive the kinematic transformations that relates the coordinates used in these two areas. Details of these derivations are often omitted in the literature, and blunders are common. The second objective is to introduce a notation which is consistent with the coordinates used in both areas.

2 Reference Frames

To describe the position, orientation, forces and geometry of a ship the following four orthonormal right-hand-sided reference frames are commonly used—see Figures 1:

- Geometric (g-frame),
- Noth-east-down (n-frame),
- Body-fixed (b-frame),
- Hydrodynamic (h-frame).

These frames have specific uses:

- The g-frame is used to define the geometry of the hull (table of offsets), the floating conditions (heel and trim), and the location of other reference frames in the vessel.
- The n-frame is used to define the position of the vessel on the earth, and the direction of wind and current.
- The b-frame is the frame to which all the velocity and acceleration measurements taken on board are referred. This frame is also used to formulate the equations of motion and to define some ship motion performance indices.
- The h-frame is used to define the wave elevation at the vessel location and to compute some of the hydrodynamic forces and parameters using standard hydrodynamic programs.

The location of the different reference frames are defined as follows (Perez, 2005):

- The ***g*-frame** $(o_g, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ is fixed to the hull. The positive unit vector $\mathbf{g}_1$ along the x-axis points towards the bow, $\mathbf{g}_2$ along the y-axis points towards starboard and $\mathbf{g}_3$ along the z-axis points upwards. The origin of this frame o_g is located along the centre-line plane and at the intersection of the baseline (BL) and the aft perpendicular (AP), which is usually taken at the rudder stock—see Figure 2.

- The ***n*-frame** $(o_n, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ is a local geographical frame fixed to the Earth. The positive unit vector $\mathbf{n}_1$ points towards the North, $\mathbf{n}_2$ points towards the East, and $\mathbf{n}_3$ points towards the centre of the Earth. The origin, o_n, is located on mean water free-surface at an appropriate location. This frame is considered inertial. This is a reasonable

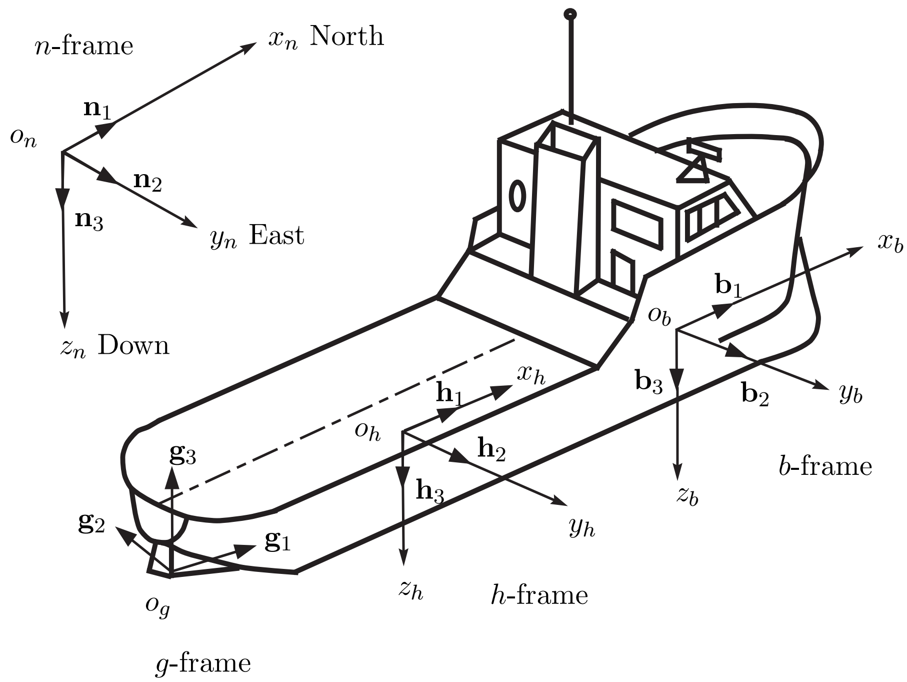

Figure 1: Notation and sign conventions for ship motion description.

assumption because the velocity of marine vehicles is small enough for the forces due to the rotation of the Earth being negligible compared to the hydrodynamic forces acting on the vehicle (Fossen, 2002).

- The ***b*-frame** $(o_b, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ is fixed to the hull. The positive unit vector $\mathbf{b}_1$ points towards the bow, $\mathbf{b}_2$ points towards starboard and $\mathbf{b}_3$ points downwards. For marine vehicles, the axes of this frame are chosen to coincide with the principal axes of inertia; this determines the position of the origin of the frame, o_b, (Fossen, 2002).
- The ***h*-frame**$(o_h, \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$ is not fixed to the hull; it moves at the average speed of the vessel following its path. The positive unit vector $\mathbf{h}_1$ points forward and it is aligned with the low-frequency heading angle $\overline{\psi}$ [1]. The positive unit vector $\mathbf{h}_2$ points towards starboard, and $\mathbf{h}_3$ points downwards. The horizontal plane that contains the unit vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ coincides with the mean free surface of the water.

[1] The angle $\bar{\psi}$ is obtained by filtering out the 1st-order wave-induced motion (oscillatory motion), and keeping the low frequency motion, which can be either equilibrium or slowly-varying. Hence, $\bar{\psi}$ is constant for a ship sailing in a straight-line path.

The origin o_h is determined such that the z_h-axis passes through the centre of gravity. This frame is considered when the vessel sails at a constant speed (which also includes the case of zero speed), and a constant heading; and therefore, this frame is considered inertial.

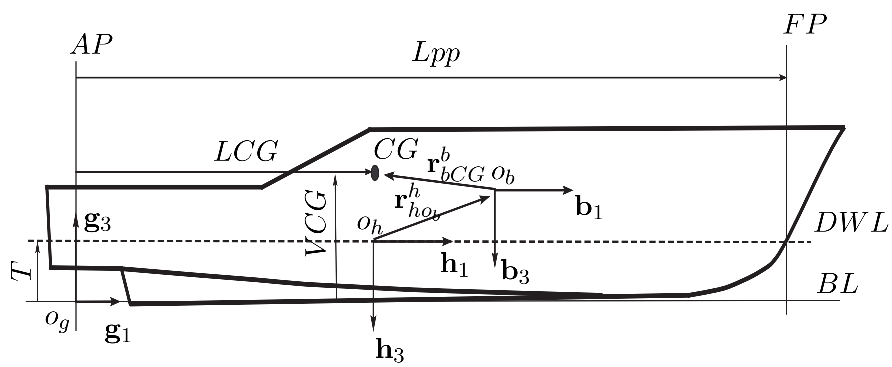

Figure 2: Main particulars and reference frames: geometric (origin o_g); hydrodynamic (origin o_h); and body-fixed (origin o_b); CG—centre of gravity; LCG—lateral centre of gravity (distance); VCG—vertical centre of gravity (distance); AP—aft perpendicular; FP—front perpendicular; L_{pp}—length between perpendiculars; T—draught; DWL—design waterline and BL–baseline.

The positive convention for the different frames described above will be adopted in the rest of the report, and it is also the positive convention adopted in Marine Systems Simulator (MSS, 2004; Smogeli *et al.*, 2005). In the literature and in different hydrodynamic programs, however, you may find other conventions.

3 Mathematical Notation for Vectors

To describe vectors we will use cartesian coordinate frames, and the coordinates will be denoted by a column vector. Therefore, a vector $\mathbf{u}$ can be described as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T. \tag{1}$$

The vector $\mathbf{u}$ can be considered independent of a reference frame, and in this case it is referred to as a *free vector*—this is useful when introducing relationships between vectors (*e.g.*, the cross product) that hold independent of the reference in which the vectors are expressed.

When a vector is described relative to a frame f, however, we will use the following notation:

$$\mathbf{u}^f = \begin{bmatrix} u_1^f \\ u_2^f \\ u_3^f \end{bmatrix} = \begin{bmatrix} u_1^f & u_2^f & u_3^f \end{bmatrix}^T. \tag{2}$$

To describe the motion of the ship, we use *bound vectors*, *i.e.*, vectors for which the *line of action* and the *point of application* are both prescribed in addition to its direction and modulus (Sciavicco and Siciliano, 2004). Then, it is convenient to use a mathematical notation that indicates the particular application point and the reference frame in which the vector is expressed.

The *position* of a point s with respect to the reference frame f, expressed in the frame g will be denoted by

$$\mathbf{r}_{fs}^g = x_{fs}^g \mathbf{g}_1 + y_{fs}^g \mathbf{g}_2 + z_{fs}^g \mathbf{g}_3 \tag{3}$$

where $\mathbf{g}_i$, $i = 1, 2, 3$ are the unit vectors along the reference g-frame axes, and x_{fs}^g, y_{fs}^g and z_{fs}^g are the components of the vector $\mathbf{r}_{fs}^g$ in this frame. As a short-hand notation, we will use the *coordinate vector* form

$$\mathbf{r}_{fs}^g = \begin{bmatrix} x_{fs}^g \\ y_{fs}^g \\ z_{fs}^g \end{bmatrix} = [x_{fs}^g, y_{fs}^g, z_{fs}^g]^{\mathrm{T}}. \tag{4}$$

Note, for example, the use of this notation in Figure 2 to indicate the position of the centre of grabity CG with respect to the b-frame, *i.e.*, $\mathbf{r}_{bCG}^b$.

A similar notation will be used for the velocities and accelerations:

- $\mathbf{v}_{fs}^g$ denotes the *velocity* of the point s with respect to a frame f, expressed in the frame g, *i.e.*, $\mathbf{v}_{fs}^g = \dot{\mathbf{r}}_{fs}^g$.
- $\dot{\mathbf{v}}_{fs}^g$ denotes the *acceleration* of the point s with respect to a frame f, expressed in the frame g, *i.e.*, $\dot{\mathbf{v}}_{fs}^g = \ddot{\mathbf{r}}_{fs}^g$.

When we consider the velocity of rotation of one reference frame with respect to another, we will denote this by

- $\boldsymbol{\omega}_{fg}^g$ and $\boldsymbol{\omega}_{fg}^f$, which are the angular velocity vectors of the frame g with respect to the frame f expressed in the frame g and f respectively.

The definition of angular velocities will be addressed in more details latter.

4 Coordinates for Ship Motion Description

4.1 Manoeuvring Coordinates

The *position* of a vessel is defined by the coordinates of the origin of the b-frame, o_b, relative to the n-frame:

$$\boxed{\mathbf{r}^n_{no_b} \triangleq \begin{bmatrix} n \\ e \\ d \end{bmatrix},}$$

where n denotes the *North* component, e denotes the *East* component, and d denotes the *Down* component. This is illustrated in Figure 3, which also shows the ship trajectory and the linear-velocity vector $\dot{\mathbf{r}}^n_{o_b}$. The *attitude*

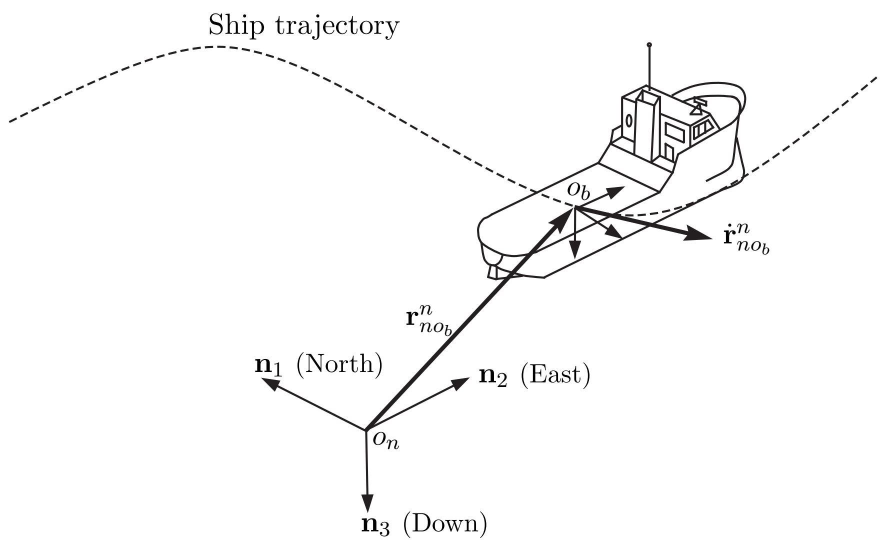

Figure 3: Ship position and velocity vectors.

(or *orientation*) of a vessel is defined by the orientation of the b-frame relative to the n-frame. This is given by the *three* consecutive rotations about the main axes that take the n-frame into the b-frame. These rotations can be performed in a different order (there are 12 different ways of doing this), and each triplet of angles rotated are called a set of *Euler angles*. The most commonly used set of Euler angles are *yaw*, *pitch* and *roll*, which correspond the rotations performed in the following order—see Figure 4:

1. rotation about the z_n axis an angle ψ (yaw angle),

2. rotation about the y' axis an angle θ (pitch angle),

3. rotation about the x'' axis an angle ϕ (roll angle).

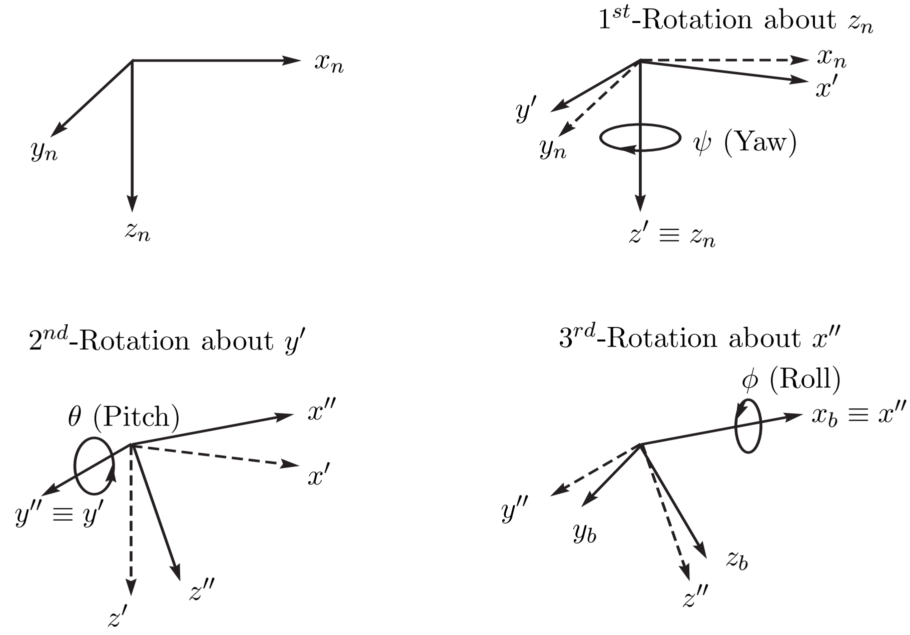

Figure 4: Yaw, pitch and Roll.

The vector of roll, pitch and yaw that take the n-frame into the orientation of the b-frame using will be defined[2] as

$$\boxed{\mathbf{\Theta}_{nb} \triangleq \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}. \qquad (5)}$$

Following the notation of Fossen (1994; 2002), the *generalised position vector* (position and orientation) is defined as

$$\boxed{\boldsymbol{\eta} \triangleq \begin{bmatrix} \mathbf{r}^n_{no_b} \\ \mathbf{\Theta}_{nb} \end{bmatrix} = [n, e, d, \phi, \theta, \psi]^{\mathrm{T}}. \qquad (6)}$$

The linear and angular velocities of the ship are more conveniently expressed in the b-frame—this simplifies the equations of motion and is consistent with the measurements taken onboard (Fossen, 1994; Fossen, 2002).

[2] The order in (5) is consistent with the way in which the equations of motion of marine vehicles are written in the literature. It should be noted, however, that the rotations are performed in reverse order, *i.e.*, $\psi \to \theta \to \phi$.

The *generalised velocity vector* (linear-angular velocity vector) given in the b-frame is defined as:

$$\boxed{\boldsymbol{\nu} \triangleq \begin{bmatrix} \mathbf{v}^b_{no_b} \\ \boldsymbol{\omega}^b_{nb} \end{bmatrix} = [u, v, w, p, q, r]^{\mathrm{T}}, \qquad (7)}$$

where according to the adopted notation:

- $\mathbf{v}^b_{no_b} = [u, v, w]^{\mathrm{T}}$ is the linear velocity of the point o_b with respect to the n-frame expressed in the b-frame. Thus, $\mathbf{v}^b_{no_b}$ results from expressing $\dot{\mathbf{r}}^n_{no_b}$ in the b-frame—see Figure 3. In Section 7, we will discuss the kinematic transformation that relates these two velocity vectors.

- $\boldsymbol{\omega}^b_{nb} = [p, q, r]^{\mathrm{T}}$ is the angular velocity of the b-frame with respect to the n-frame expressed in the frame b.

4.2 Seakeeping Coordinates

As mentioned in the introduction, in seakeeping theory the study ship motion is performed under the assumption that the ship is motion can be described as the superposition of a constant and a zero-mean oscillatory components. The oscillatory motion is due to the first order wave excitation. Note that the case of zero forward speed is also contemplated.

The constant course and speed define a *state of equilibrium* of motion, and the action of the waves makes the ship oscillate with respect to this equilibrium—the oscillations may not necessarily be harmonic. This fundamental assumption is the basis of the *seakeeping theory of ship motion*, and upon this rely different computational methods used to predict the ship motion and induced loads. This assumption results in the h-frame being considered inertial; and therefore, the equations of motion in this frame are linear and the frequency domain approach can be used to calculate forces and moments due to sinusoidal wave excitations—see (Bertram, 2004; Faltinsen, 1990; Lloyd, 1989; Newman, 1977).

In the absence of wave excitation, the origin o_h coincides with the location of a point s in the ship. Under the action of the waves, the hull is disturbed from its equilibrium and the point s oscillates, with respect to its equilibrium position. This is illustrated in Figure 5.

The *Seakeeping coordinates* are the following set of *generalised perturbation coordinates*:

$$\boxed{\boldsymbol{\xi} \triangleq \begin{bmatrix} \mathbf{r}^h_{hs} \\ \boldsymbol{\Theta}_{hs} \end{bmatrix} = [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6]^{\mathrm{T}}. \qquad (8)}$$

Thus, the first three coordinates describe the position of the point s with respect to the h-frame, and the last three coordinates are the Euler angles

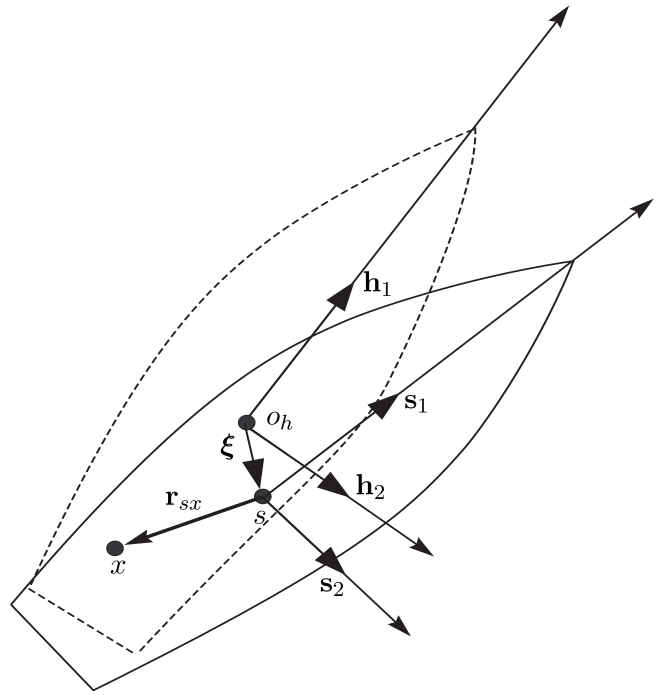

Figure 5: Seakeeping coordinates. In absence of wave excitation, the h- and s-frame coincide.

that take the h-frame into the orientation of the s-frame $(s, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$ fixed to the body at at the point s—see Figure 5.

The linear coordinates of (8) are referred to as

- ξ_1—surge perturbation,
- ξ_2—sway perturbation,
- ξ_3—heave perturbation,

If the axes of the s-frame are parallel to those of the b-frame, we can then write

$$\boldsymbol{\Theta}_{hs} = \boldsymbol{\Theta}_{hb} \triangleq \begin{bmatrix} \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix} = \begin{bmatrix} \phi \\ \theta \\ \psi - \overline{\psi} \end{bmatrix}, \tag{9}$$

and these angles are referred to as

- ξ_4—roll perturbation,
- ξ_5—pitch perturbation,

- ξ_6—yaw perturbation.

The perturbation coordinates can be used to describe the oscillatory position of any point of interest with respect to h-frame. Indeed, if the coordinates $\boldsymbol{\xi}$ describe the motion of the point s in the hull, and we are interested in the motion of point of interest x, then following relationships hold for the velocities and accelerations:

$$\begin{aligned}\mathbf{v}_{hx}^{h} &= [\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3]^{\mathrm{T}} + [\dot{\xi}_4, \dot{\xi}_5, \dot{\xi}_6]^{\mathrm{T}} \times \mathbf{r}_{sx}^{h} \\ \dot{\mathbf{v}}_{hx}^{h} &= [\ddot{\xi}_1, \ddot{\xi}_2, \ddot{\xi}_3]^{\mathrm{T}} + [\ddot{\xi}_4, \ddot{\xi}_5, \ddot{\xi}_6]^{\mathrm{T}} \times \mathbf{r}_{sx}^{h}.\end{aligned} \tag{10}$$

Where the vector $\mathbf{r}_{sx}^{h}$ is the position of the point x with respect to the s-frame fixed at the point s in the hull, and expressed in the h-frame—see Figure 5. To express the position of x in the h-frame, we need to introduce rotation matrices; we will defer this to Section 6.

The cross-product can be conveniently expressed in a matrix form as (Egeland and Gravdahl, 2002; Fossen, 2002):

$$\mathbf{a} \times \mathbf{b} \triangleq \mathbf{S}(\mathbf{a})\mathbf{b} = -\mathbf{b} \times \mathbf{a} = -\mathbf{S}(\mathbf{b})\mathbf{a}, \tag{11}$$

where the *skew-symmetric matrix operator* $\mathbf{S}(\cdot)$ is defined as:

$$\boxed{\mathbf{S} : \mathbb{R}^3 \mapsto \mathbb{R}^{3\times 3}, \mathbf{S}(\mathbf{x}) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \quad (12)}$$

Note that $\mathbf{S}(\mathbf{a}) = -\mathbf{S}^{\mathrm{T}}(\mathbf{a})$.

Using this, we can re-write (10) as

$$\boxed{\begin{aligned}\mathbf{v}_{hx}^{h} &= [\mathbf{I}_{3\times 3},\, \mathbf{S}^{\mathrm{T}}(\mathbf{r}_{sx}^{h})]\dot{\boldsymbol{\xi}}, \\ \dot{\mathbf{v}}_{hx}^{h} &= [\mathbf{I}_{3\times 3},\, \mathbf{S}^{\mathrm{T}}(\mathbf{r}_{sx}^{h})]\ddot{\boldsymbol{\xi}}.\end{aligned} \quad (13)}$$

The validity of expressions (10) and (13) can be verified using the results of Section 6.

4.3 Seakeeping Coordinates in Hydrodynamic Programmes

Different hydrodynamic programmes can output data in different forms. Usually, the variables $\boldsymbol{\xi}$ are defined as in (8). Some codes, however, allow the user to specify the point at which the output data is calculated, say x; typically this point is chosen to be the centre of gravity CG, or any other point of interest where the motion needs to be evaluated. When this is done, the output of the programme is the perturbation of x, with respect to its equilibrium position $\bar{x}$ (no wave excitation):

$$\delta\boldsymbol{\xi}_x = x - \bar{x} = \begin{bmatrix} [\mathbf{I}_{3\times3} & \mathbf{S}^{\mathrm{T}}(\mathbf{r}^s_{sx})] \\ [\mathbf{0}_{3\times3} & \mathbf{I}_{3\times3}] \end{bmatrix} \boldsymbol{\xi}. \tag{14}$$

Where the vector $\mathbf{r}^s_{sx}$ is the position of the point x with respect to the s-frame expressed in the s-frame—see Figure 5. That is, $\delta\boldsymbol{\xi}_x$ are the generalised coordinates of x with respect to a frame fixed at the equilibrium position $\bar{x}$ and not fixed to the vessel. The above formula is used to express the RAO at the location x. Note that the velocities and accelerations are easily obtained from (14) since $\mathbf{r}^s_{sx}$ is time invariant.

4.4 Unified Coordinates

It follows from the discussion thus far that the seakeeping coordinates are valid only for a very specific set of conditions that render the h-frame inertial, *i.e.*, constant speed and slowly-varying changes in heading. Despite these restrictions, seakeeping coordinates are very important because most of the theory of ship hydrodynamics and the codes used to compute hydrodynamic data (motion and loads) use these coordinates. The use of these data for preliminary ship design is nowadays standard; and therefore, it can be used to develop models useful for control and fault diagnosis.

If we seek a set of coordinates to describe ship motion that are valid for all the vessel operational conditions, we need to use the *manoeuvring generalised coordinates* $\boldsymbol{\eta}$ and $\boldsymbol{\nu}$ defined in (6) and (7). This set of coordinates have the following properties (Fossen, 2002):

- These coordinates describe the motion in any regime (no restrictions to constant course or speed).
- Velocities and accelerations measurements taken on board are conveniently expressed in body-fixed frames; therefore, the manoeuvring coordinates are the natural choice.
- The equations of motion in terms of these coordinates are simple to implement for computer simulation.

Table 1 sumarises the notation of the manoeuvring and seakeeping coordinates. Before moving on to the kinematic transformations that express the relations between velocities in the different frames, let us review in the next section additional angular coordinates used in guidance of marine systems.

Table 1: Summary of nomenclature used for the manoeuvring and seakeeping coordinates.

Coordinate	**Name**	**Definition frame**
n	North position	n-frame
e	East position	n-frame
d	Down position	n-frame
ϕ	Roll angle	Euler angle $(n \to b)$
θ	Pitch angle	Euler angle $(n \to b)$
ψ	Heading or yaw angle	Euler angle $(n \to b)$
u	Surge velocity	b-frame
v	Sway velocity	b-frame
w	Heave velocity	b-frame
p	Roll rate	b-frame
q	Pitch rate	b-frame
r	Yaw rate	b-frame
ξ_1	Surge perturbation	h-frame
ξ_2	Sway perturbation	h-frame
ξ_3	Heave perturbation	h-frame
ξ_4	Roll perturbation	Euler angle $(h \to b)$
ξ_5	Pitch perturbation	Euler angle $(h \to b)$
ξ_6	Yaw perturbation	Euler angle $(h \to b)$

5 Additional Angular Coordinates Used in Guidance

Let us define the *total ship-velocity vector* in the b-frame:

$$\mathbf{v}^b_{no_b} = [u, v, w]^{\mathrm{T}}. \tag{15}$$

This velocity vector can be separated into its slowly-varying or equilibrium components and perturbations about these components:

$$\begin{aligned} u &= \bar{u} + \delta u \\ v &= \bar{v} + \delta v \\ w &= \bar{w} + \delta w. \end{aligned}$$

Then, for the angles about the z-axis of surface ships, it is convenient to distinguish between the following (see Figure 6):

- **Heading or yaw angle** ψ. This is the first rotation of the sequence of rotations (Euler angles) that take the n- into the b-frame—see Section 4.1.

- **Yaw perturbation angle** ξ_6. This is the first rotation of the sequence of rotations (Euler angles) that take the h- into the b-frame—see Section 4.2.

- **Drift angle** β. This is the angle between the positive x-axis of the b-frame and the average ship velocity vector $\bar{\mathbf{u}}$, *i.e.*,

$$\beta = \arctan\left(\frac{\bar{v}}{\bar{u}}\right), \tag{16}$$

provided $\bar{u}$ is not zero.

- **Course angle** γ. This is the angle between the positive x-axis of the n-frame the ship velocity vector $\bar{\mathbf{u}}$.

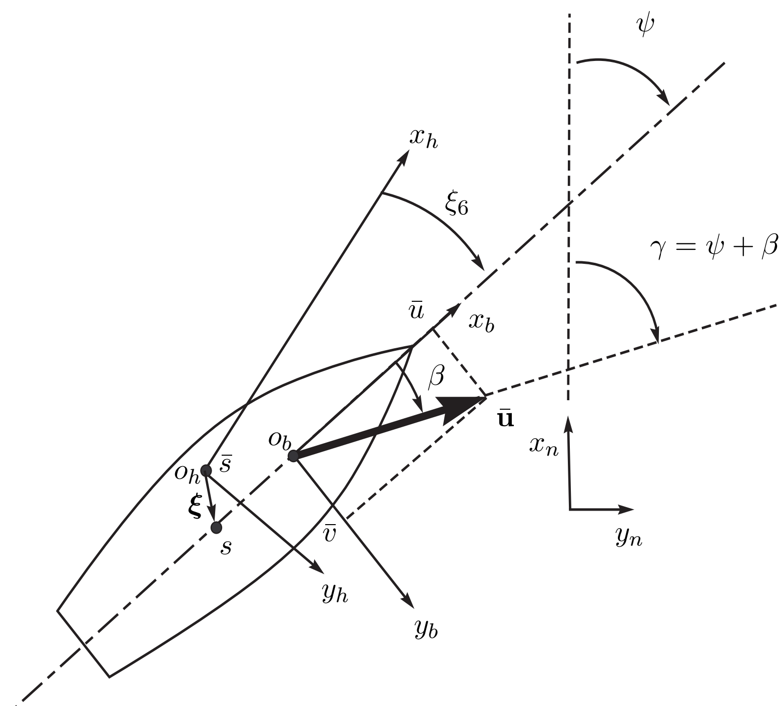

Figure 6: Angles for the horizontal plane.

6 Kinematic Transformations

6.1 Rotation Matrices

The transformation of vector coordinates between different frames is performed via appropriate transformation matrices. Following (Egeland and

Gravdahl, 2002), the generic unbound vector $\mathbf{r}$, can be expressed in either the frame a or the frame b as

$$\mathbf{r} = \sum_{i=1}^{3} r_i^a \mathbf{a}_i \quad \text{and} \quad \mathbf{r} = \sum_{i=1}^{3} r_i^b \mathbf{b}_i, \tag{17}$$

where the vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ are the unit vectors along the axis of the reference frames a and b respectively, and $r_i^a = \mathbf{r} \cdot \mathbf{a}_i$ and $r_i^b = \mathbf{r} \cdot \mathbf{b}_i$. Then,

$$r_i^a = \mathbf{r} \cdot \mathbf{a}_i = \left(\sum_{j=1}^{3} r_j^b \mathbf{b}_j \right) \cdot \mathbf{a}_i = \sum_{j=1}^{3} r_j^b (\mathbf{a}_i \cdot \mathbf{b}_j). \tag{18}$$

In matrix form,

$$\underbrace{\begin{bmatrix} r_1^a \\ r_2^a \\ r_3^a \end{bmatrix}}_{\mathbf{r}^a} = \underbrace{\begin{bmatrix} (\mathbf{a}_1 \cdot \mathbf{b}_1) & (\mathbf{a}_1 \cdot \mathbf{b}_2) & (\mathbf{a}_1 \cdot \mathbf{b}_3) \\ (\mathbf{a}_2 \cdot \mathbf{b}_1) & (\mathbf{a}_2 \cdot \mathbf{b}_2) & (\mathbf{a}_2 \cdot \mathbf{b}_3) \\ (\mathbf{a}_3 \cdot \mathbf{b}_1) & (\mathbf{a}_3 \cdot \mathbf{b}_2) & (\mathbf{a}_3 \cdot \mathbf{b}_3) \end{bmatrix}}_{\mathbf{R}_b^a} \underbrace{\begin{bmatrix} r_1^b \\ r_2^b \\ r_3^b \end{bmatrix}}_{\mathbf{r}^b}, \tag{19}$$

from which it follows the notation $\mathbf{R}_b^a$ for the so-called *rotation matrix* from a to b (which takes vectors expressed in the frame b to the frame a):

$$\boxed{\mathbf{R}_b^a \triangleq \begin{bmatrix} (\mathbf{a}_1 \cdot \mathbf{b}_1) & (\mathbf{a}_1 \cdot \mathbf{b}_2) & (\mathbf{a}_1 \cdot \mathbf{b}_3) \\ (\mathbf{a}_2 \cdot \mathbf{b}_1) & (\mathbf{a}_2 \cdot \mathbf{b}_2) & (\mathbf{a}_2 \cdot \mathbf{b}_3) \\ (\mathbf{a}_3 \cdot \mathbf{b}_1) & (\mathbf{a}_3 \cdot \mathbf{b}_2) & (\mathbf{a}_3 \cdot \mathbf{b}_3) \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1^a & \mathbf{b}_2^a & \mathbf{b}_1^a \end{bmatrix}. \quad (20)}$$

As indicated above, the entries of a rotation matrix are the inner products of the unit vectors of the reference frames involved:

$$\mathbf{R}_b^a(i,j) = (\mathbf{a}_i \cdot \mathbf{b}_j), \qquad i = 1,2,3; \quad j = 1,2,3. \tag{21}$$

We can also see that the columns of the rotation matrix are the unit vectors of the b-frame expressed in the a-frame.

Rotation matrices have the following properties:

1. They can be seen as transformation matrices. This takes the coordinates of the vector $\mathbf{r}$ in the b-frame to the coordinates in the a-frame, *i.e.*,
$$\mathbf{r}^a = \mathbf{R}_b^a \, \mathbf{r}^b. \tag{22}$$

2. They can be seen as rotation matrices. If the vector $\mathbf{r}$ has coordinates $\mathbf{r}^a$ is rotated to the vector $\mathbf{s}$ with coordinates $\mathbf{s}^b = \mathbf{r}^a$, then
$$\mathbf{s}^a = \mathbf{R}_b^a \, \mathbf{r}^a. \tag{23}$$

3. They are elements of the *special orthogonal group of order 3*, $SO(3)$:

$$SO(3) = \left\{\mathbf{R}|\mathbf{R} \in \mathbb{R}^{3\times3},\ \mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}_{3\times3}, \text{ and} \det(\mathbf{R}){=}1\right\}. \tag{24}$$

   Thus,

$$(\mathbf{R}_b^a)^{-1} = (\mathbf{R}_b^a)^{\mathrm{T}} = \mathbf{R}_a^b.$$

6.2 Composite and Simple Rotations

The rotation from a frame a to a frame c can be describes as a composite rotation from the frame a to the frame b and then a rotation from the frame b to the frame c:

$$\begin{aligned} \mathbf{r}^b &= \mathbf{R}_c^b\,\mathbf{v}^c \\ \mathbf{r}^a &= \mathbf{R}_b^a\,\mathbf{v}^b \end{aligned} \tag{25}$$

Combining the above, we obtain

$$\mathbf{r}^a = \mathbf{R}_b^a\,\mathbf{R}_c^b\,\mathbf{v}^c. \tag{26}$$

Then,

$$\boxed{\mathbf{R}_c^a = \mathbf{R}_b^a\,\mathbf{R}_c^b. \qquad (27)}$$

This result is extendable to any number of intermediate rotations.

A rotation is *simple*, if it is a rotation about a single axis (not necessary a coordinate axis). If we consider rotations about the coordinates axes we obtain:

- A rotation of an angle ψ about the z-axis:

$$\mathbf{R}_{z,\psi} \triangleq \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{28}$$

- A rotation of an angle θ about the y-axis:

$$\mathbf{R}_{y,\theta} \triangleq \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{29}$$

- A rotation of an angle ϕ about the x-axis:

$$\mathbf{R}_{x,\phi} \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \tag{30}$$

The above results follow readily from (20). For example, the single rotation about the z-axis (28) can be obtained by calculating the inner products of the unit vectors shown in Figure Figure 7:

$$\begin{aligned}
(\mathbf{a}_1 \cdot \mathbf{b}_1) &= \cos\psi \\
(\mathbf{a}_1 \cdot \mathbf{b}_2) &= -\sin\psi \\
(\mathbf{a}_2 \cdot \mathbf{b}_1) &= \sin\psi \\
(\mathbf{a}_2 \cdot \mathbf{b}_2) &= \cos\psi \\
(\mathbf{a}_1 \cdot \mathbf{b}_3) &= (\mathbf{a}_2 \cdot \mathbf{b}_3) = (\mathbf{a}_3 \cdot \mathbf{b}_1) = (\mathbf{a}_3 \cdot \mathbf{b}_2) = 0 \\
(\mathbf{a}_3 \cdot \mathbf{b}_3) &= 1.
\end{aligned} \tag{31}$$

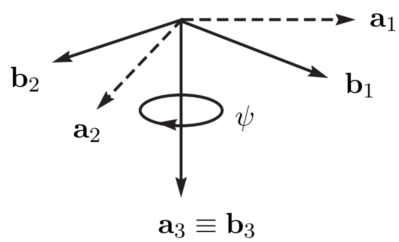

Figure 7: Single rotation about the z-axis.

6.3 Rotation Matrices in Terms of Yaw, Pitch and Roll

The rotation matrix can be described in different ways and as a function of different parameters—see, for example, Egeland and Gravdahl (2002). When we discussed the orientation of the vessel in Section 4.1, we did in terms of yaw, pitch and yaw—a particular choice of Euler angles. In this section, we will express the rotation matrix in terms of these angles.

Using the simple rotation matrices (28)-(30), and the extension of expression (27) for the three composite rotations, we obtain:

$$\boxed{\mathbf{R}_b^a = \mathbf{R}_{z,\psi}\,\mathbf{R}_{y,\theta}\,\mathbf{R}_{x,\phi}. \qquad (32)}$$

Note that the matrix multiplication order above is consistent with the fact that $\mathbf{v}^a = \mathbf{R}_b^a\,\mathbf{v}^b$.

After performing the matrix multiplications, we find *the rotation matrix in terms of yaw, pitch and roll* is

$$\mathbf{R}_b^a(\mathbf{\Theta}_{ab}) = \begin{bmatrix} c\psi c\theta & -s\psi c\phi + c\psi s\theta s\phi & s\psi s\phi + c\psi c\phi s\theta \\ s\psi c\theta & c\psi c\phi + s\phi s\theta s\psi & -c\psi s\phi + s\psi c\phi s\theta \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}, \quad (33)$$

where $s \equiv \sin(\cdot)$ and $c \equiv \cos(\cdot)$. To make explicit that we are using yaw, pitch and roll to express the rotation matrix, we have used the notation $\mathbf{R}_b^a(\mathbf{\Theta}_{ab})$, where $\mathbf{\Theta}_{ab}$ is the vector of yaw, pitch and roll that take the a-frame into the orientation of the b-frame:

$$\mathbf{\Theta}_{ab} \triangleq \begin{bmatrix} \phi, & \theta, & \psi \end{bmatrix}^T. \quad (34)$$

7 Velocity Transformations Between the b- and n-frame

In this section, we will study the transformation that relates the the body-fixed generalised velocity vector $\boldsymbol{\nu}$—see (7)—to the time derivative of the generalised position vector $\boldsymbol{\eta}$—see (6).

7.1 Linear-velocity Transformation

As illustrated in Figure 3, the position of the ship is given by the vector $\mathbf{r}_{no_b}^n$, and the position trajectory is given by

$$\mathbf{v}_{no_b}^n = \dot{\mathbf{r}}_{no_b}^n = \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{bmatrix} = \begin{bmatrix} \dot{n} \\ \dot{e} \\ \dot{d} \end{bmatrix}. \quad (35)$$

The linear velocity vector in the b-frame is simply obtained by expressing $\mathbf{v}_{no_b}^n$ in the b-frame. Therefore, the *linear velocity transformation between the b- and the n-frame* is simply a rotation:

$$\mathbf{v}_{no_b}^n = \mathbf{R}_b^n(\mathbf{\Theta}_{nb})\,\mathbf{v}_{no_b}^b. \quad (36)$$

Component wise,

$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{bmatrix} = \mathbf{R}_b^n(\mathbf{\Theta}_{nb}) \begin{bmatrix} \dot{\nu}_1 \\ \dot{\nu}_2 \\ \dot{\nu}_3, \end{bmatrix} \quad (37)$$

with $\mathbf{R}_b^n(\mathbf{\Theta}_{nb})$ given by (33) (modulo substitution a by n).

7.2 Angular-velocity Transformation

Similar to the previous section, we would like to find the transformation that relates the time derivatives of yaw, pitch and roll to the body-fixed angular velocities r, q and p. This transformation is a little more involved than that of the linear velocities.

We start by noticing that the rotation matrix $\mathbf{R}_b^a$ is orthogonal, *i.e.*, $\mathbf{R}_b^a(\mathbf{R}_b^a)^T = \mathbf{I}$. Therefore,

$$\frac{d}{dt}[\mathbf{R}_b^a(\mathbf{R}_b^a)^T] = \dot{\mathbf{R}}_b^a(\mathbf{R}_b^a)^T + \mathbf{R}_b^a(\dot{\mathbf{R}}_b^a)^T = \mathbf{0}. \tag{38}$$

The above expression implies that $\dot{\mathbf{R}}_b^a(\mathbf{R}_b^a)^T$ is a skew-symmetric matrix. As mentioned by Egeland and Gravdahl (2002), this means that it can be described by a column vector.

The vector $\boldsymbol{\omega}_{ab}$ of angular velocity of the frame b with respect to the frame a, with coordinates in the frame a satisfies

$$\boxed{\mathbf{S}(\boldsymbol{\omega}_{ab}^a) = \dot{\mathbf{R}}_b^a(\mathbf{R}_b^a)^T, \tag{39}}$$

where $\mathbf{S}(\cdot)$ is the skew-symmetric matrix operator defined in (12). It also follows that

$$\boxed{\dot{\mathbf{R}}_b^a = \mathbf{S}(\boldsymbol{\omega}_{ab}^a)\,\mathbf{R}_b^a = \mathbf{R}_b^a\,\mathbf{S}(\boldsymbol{\omega}_{ab}^b). \tag{40}}$$

Using the simple rotation matrices (28)–(30), we can define angular velocities for simple rotations as

$$\begin{aligned}
\boldsymbol{\omega}_{z,\dot{\psi}}^a &= \dot{\mathbf{R}}_{z,\psi}(\mathbf{R}_{z,\psi})^T \\
\boldsymbol{\omega}_{y,\dot{\theta}}^a &= \dot{\mathbf{R}}_{y,\theta}(\mathbf{R}_{y,\theta})^T \\
\boldsymbol{\omega}_{x,\dot{\phi}}^a &= \dot{\mathbf{R}}_{x,\phi}(\mathbf{R}_{x,\phi})^T.
\end{aligned} \tag{41}$$

After taking the derivatives and multiplying we obtain

$$\boldsymbol{\omega}_{z,\dot{\psi}}^a = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \quad \boldsymbol{\omega}_{y,\dot{\theta}}^a = \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} \quad \boldsymbol{\omega}_{x,\dot{\phi}}^a = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}. \tag{42}$$

To obtain insight into the meaning of $\boldsymbol{\omega}_{ab}$ and the particular ones above, we can use the angle-axis parameterisation of the rotation matrix:

$$\mathbf{R}_b^a = \mathbf{R}_{\alpha,\mathbf{k}^a} = \mathbf{I} + \mathbf{S}(\mathbf{k}^a)\sin\alpha + \mathbf{S}(\mathbf{k}^a)\mathbf{S}(\mathbf{k}^a)(1-\cos\alpha), \tag{43}$$

where $\mathbf{k}$ is a constant unit vector along the axis of rotation, and α is the rotated angle. This representation uses a single rotation instead of the three rotations described in Section 6.3. By combining (43) with (39) we can obtain the following relationship—see Egeland and Gravdahl (2002):

$$\begin{aligned}
\mathbf{S}(\boldsymbol{\omega}_{ab}^a) &= \dot{\alpha}(\mathbf{S}(\mathbf{k}^a)\cos\alpha + \mathbf{S}(\mathbf{k}^a)\mathbf{S}(\mathbf{k}^a)\sin\alpha) \\
&\quad (\mathbf{I} - \mathbf{S}(\mathbf{k}^a)\sin\alpha + \mathbf{S}(\mathbf{k}^a)\mathbf{S}(\mathbf{k}^a)(1-\cos\alpha)) \\
&= \dot{\alpha}\left[\mathbf{S}(\mathbf{k}^a)\cos\alpha + \mathbf{S}(\mathbf{k}^a)^2\sin\alpha - \mathbf{S}(\mathbf{k}^a)^2\cos\alpha\sin\alpha \right. \\
&\quad + \mathbf{S}(\mathbf{k}^a)\sin^2\alpha - \mathbf{S}(\mathbf{k}^a)(\cos\alpha - \cos^2\alpha) + \mathbf{S}(\mathbf{k}^a)^2\cos\alpha\sin\alpha \\
&\quad \left. -\mathbf{S}(\mathbf{k}^a)^2\sin\alpha\right] \\
&= \dot{\alpha}\,\mathbf{S}(\mathbf{k}^a),
\end{aligned} \tag{44}$$

which implies

$$\boxed{\boldsymbol{\omega}_{ab}^a = \dot{\alpha}\,\mathbf{k}^a.} \tag{45}$$

This is in agreement with expressions (42), and is valid provided the direction of rotation ($\mathbf{k}$) is stationary.

If we consider now a three-stage composite rotation $\mathbf{R}_d^a = \mathbf{R}_b^a\mathbf{R}_c^b\mathbf{R}_d^c$, its time derivative satisfies

$$\dot{\mathbf{R}}_d^a = \dot{\mathbf{R}}_b^a\mathbf{R}_c^b\mathbf{R}_d^c + \mathbf{R}_b^a\dot{\mathbf{R}}_c^b\mathbf{R}_d^c + \mathbf{R}_b^a\mathbf{R}_c^b\dot{\mathbf{R}}_d^c. \tag{46}$$

Using (39), we find that

$$\begin{aligned}
\mathbf{S}(\boldsymbol{\omega}_{ad}^a) &= \dot{\mathbf{R}}_d^a(\mathbf{R}_d^a)^T \\
&= (\dot{\mathbf{R}}_b^a\mathbf{R}_c^b\mathbf{R}_d^c + \mathbf{R}_b^a\dot{\mathbf{R}}_c^b\mathbf{R}_d^c + \mathbf{R}_b^a\mathbf{R}_c^b\dot{\mathbf{R}}_d^c)(\mathbf{R}_d^c)^T(\mathbf{R}_c^b)^T(\mathbf{R}_b^a)^T \\
&= \dot{\mathbf{R}}_b^a(\mathbf{R}_b^a)^T + \mathbf{R}_b^a\dot{\mathbf{R}}_c^b(\mathbf{R}_c^b)^T(\mathbf{R}_b^a)^T + \mathbf{R}_c^a\dot{\mathbf{R}}_d^c(\mathbf{R}_d^c)^T(\mathbf{R}_c^a)^T \\
&= \mathbf{S}(\boldsymbol{\omega}_{ab}^a) + \mathbf{R}_b^a\mathbf{S}(\boldsymbol{\omega}_{bc}^b)(\mathbf{R}_b^a)^T + \mathbf{R}_c^a\mathbf{S}(\boldsymbol{\omega}_{cd}^c)(\mathbf{R}_c^a)^T \\
&= \mathbf{S}(\boldsymbol{\omega}_{ab}^a) + \mathbf{S}(\boldsymbol{\omega}_{bc}^a) + \mathbf{S}(\boldsymbol{\omega}_{cd}^a),
\end{aligned} \tag{47}$$

which implies that

$$\boldsymbol{\omega}_{ad}^a = \boldsymbol{\omega}_{ab}^a + \boldsymbol{\omega}_{bc}^a + \boldsymbol{\omega}_{cd}^a. \tag{48}$$

This holds for any frame and not only the a-frame.

Therefore, we can say that for the composite rotation $\mathbf{R}_d^a = \mathbf{R}_b^a\mathbf{R}_c^b\mathbf{R}_d^c$ the following holds:

$$\boldsymbol{\omega}_{ad} = \boldsymbol{\omega}_{ab} + \boldsymbol{\omega}_{bc} + \boldsymbol{\omega}_{cd}. \tag{49}$$

Expression (49) is a key result to obtain the transformation we are after. Indeed, for angular-velocity transformation between the n- and the b-frame, we can write the following rotation matrix using consecutive rotations:

$$\mathbf{R}_b^n = \mathbf{R}_d^n\mathbf{R}_c^d\mathbf{R}_b^c. \tag{50}$$

This is depicted in Figure 4, where the frame d has axis (x', y', z') and the frame c has axis (x'', y'', z''). Then according to (49), the rotations (50) lead to

$$\boldsymbol{\omega}_{nb}^{b} = \boldsymbol{\omega}_{nd}^{b} + \boldsymbol{\omega}_{dc}^{b} + \boldsymbol{\omega}_{cb}^{b}. \tag{51}$$

From the latter it follows that

$$\boldsymbol{\omega}_{nb}^{b} = (\mathbf{R}_{x,\phi})^T (\mathbf{R}_{y,\theta})^T \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + (\mathbf{R}_{x,\phi})^T \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}, \tag{52}$$

which results in the sought *angular-velocity transformation transformation between the n- and the b-frame*:

$$\boldsymbol{\omega}_{nb}^{b} = \mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta}_{nb})^{-1}\, \dot{\boldsymbol{\Theta}}_{nb}, \tag{53}$$

with

$$\boldsymbol{\omega}_{nb}^{b} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \qquad \dot{\boldsymbol{\Theta}}_{nb} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{54}$$

and $\mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta}_{nb})$ and its inverse are given by

$$\begin{aligned} \mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta}_{nb}) &= \begin{bmatrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{bmatrix}, \\ \mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta}_{nb})^{-1} &= \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & c\theta s\phi \\ 0 & -s\phi & c\phi c\theta \end{bmatrix} \end{aligned} \tag{55}$$

with $s \equiv \sin(\cdot)$, $c \equiv \cos(\cdot)$, $t \equiv \tan(\cdot)$ and $c\theta \neq 0$.

Note that the transformation $\mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta}_{nb})$ is not orthogonal; therefore, $\mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta}_{nb})^T \neq \mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta}_{nb})^{-1}$.

7.3 Kinematic Transformation n-b frames

Using (36) and (53), we can define the following *kinematic transformation for the manoeuvring coordinates*:

$$\dot{\boldsymbol{\eta}} = \mathbf{J}_b^n(\boldsymbol{\Theta}_{nb})\, \boldsymbol{\nu}, \tag{56}$$

with

$$\mathbf{J}_b^n(\boldsymbol{\Theta}_{nb}) \triangleq \begin{bmatrix} \mathbf{R}_b^n(\boldsymbol{\Theta}_{nb}) & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta}_{nb}) \end{bmatrix}, \tag{57}$$

where $\mathbf{R}_b^n(\boldsymbol{\Theta}_{nb})$ given by (33) (modulo substitution a by n) and $\mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta}_{nb})$ by (55).

8 Velocity Transformations Between the h- and b-frame

In this section, we will find the kinematic transformation between the time-derivative of the generalised perturbation coordinate vector $\boldsymbol{\xi}$, *cf.*, (8), and the generalized velocity vector $\boldsymbol{\nu}$, *cf.* (7). The time-derivative of $\boldsymbol{\xi}$ is

$$\dot{\boldsymbol{\xi}} = \begin{bmatrix} \mathbf{v}_{hs}^h \\ \boldsymbol{\omega}_{hb}^h \end{bmatrix}. \tag{58}$$

To make the analysis general, we will assume that the point $s \neq o_b$. Therefore, $\dot{\boldsymbol{\xi}}$ describes the linear and angular velocity of the point s with respect to the h-frame—see Figure 5.

8.1 Linear Velocity Transformation

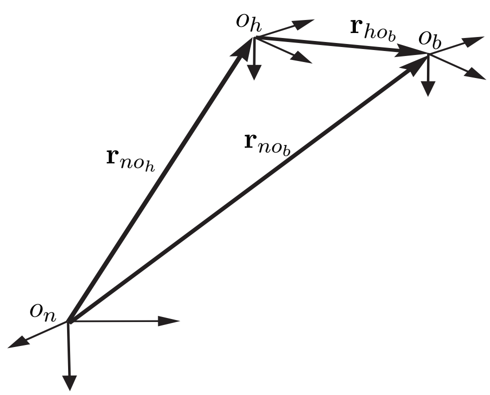

Figure 8: Relative position of frames.

Let us consider Figure 8. From this figure, it follows that independent of the coordinate frame

$$\mathbf{r}_{no_b} = \mathbf{r}_{no_h} + \mathbf{r}_{ho_b}. \tag{59}$$

Expressed in the n-frame this becomes

$$\mathbf{r}_{no_b}^n = \mathbf{r}_{no_h}^n + \mathbf{R}_h^n \mathbf{r}_{ho_b}^h, \tag{60}$$

in which

$$\mathbf{R}_h^n = \mathbf{R}_{z,\bar{\psi}} = \begin{bmatrix} \cos\bar{\psi} & -\sin\bar{\psi} & 0 \\ \sin\bar{\psi} & \cos\bar{\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{61}$$

and $\bar{\psi}$ is the slowly-varying heading of the ship that is obtained by filtering the motion induced by the first-order wave loads.

Taking the time derivative of (60), we obtain

$$\mathbf{v}^n_{no_b} = \mathbf{v}^n_{no_h} + \mathbf{R}^n_h \mathbf{v}^h_{ho_b} + \dot{\mathbf{R}}^n_h \mathbf{r}^h_{ho_b}. \tag{62}$$

which can be expresses as

$$\mathbf{v}^n_{no_b} = \mathbf{v}^n_{no_h} + \mathbf{R}^n_h \mathbf{v}^h_{ho_b} + \mathbf{S}(\boldsymbol{\omega}^n_{nh})\mathbf{R}^n_h \mathbf{r}^h_{ho_b}. \tag{63}$$

If we multiply both sides of (63) by $\mathbf{R}^b_n$, such that the left-hand side becomes $\mathbf{v}^b_{no_b}$ (the linear velocity components of $\boldsymbol{\nu}$), we obtain:

$$\mathbf{v}^b_{no_b} = \mathbf{R}^b_n \mathbf{v}^n_{no_h} + \mathbf{R}^b_h \mathbf{v}^h_{ho_b} + \mathbf{R}^b_n \mathbf{S}(\boldsymbol{\omega}^n_{nh})\mathbf{R}^n_h \mathbf{r}^h_{ho_b}. \tag{64}$$

Using (13), it follows that

$$\mathbf{v}^h_{ho_b} = [\mathbf{I}_{3\times 3} \quad \mathbf{S}^{\mathrm{T}}(\mathbf{r}^h_{so_b})]\dot{\boldsymbol{\xi}}. \tag{65}$$

where $\mathbf{r}^h_{so_b}$ is the position of the origin of the b-frame relative to the s-frame expressed in the h-frame. Substituting this in (64), we obtain

$$\boxed{\mathbf{v}^b_{no_b} = \mathbf{R}^b_n \mathbf{v}^n_{no_h} + \mathbf{R}^b_h [\mathbf{I}_{3\times 3} \quad \mathbf{S}^{\mathrm{T}}(\mathbf{r}^h_{so_b})]\dot{\boldsymbol{\xi}} + \mathbf{R}^b_n \mathbf{S}(\boldsymbol{\omega}^n_{nh})\mathbf{R}^n_h \mathbf{r}^h_{ho_b}. \quad (66)}$$

Let us make some assumptions:

Assumption 1 *The ship sails with a constant (average) heading, or the changes in heading are slow. This means we can consider* $\boldsymbol{\omega}^n_{nh} \approx \mathbf{0}$.

Assumption 2 *The vessel speed can be decomposed into two components a constant component and a perturbaton component. The constant component can be zero (anchored or DP operations). This means that* $\mathbf{v}^n_{no_h} \approx \bar{\mathbf{v}}^n_{no_h}$ or $\mathbf{v}^n_{no_h} \approx \mathbf{0}$.

Assumptions 1 and 2 result in the h-frame being inertial. Assumption 2 allows also to distinguish between transit and anchored or dynamic positioning operations. Then,

Transit:

$$\mathbf{v}^b_{no_b} = \mathbf{R}^b_n(\boldsymbol{\Theta}_{nb})\bar{\mathbf{v}}^n_{no_h} + [\mathbf{R}^b_h(\boldsymbol{\Theta}_{hb}) \quad \mathbf{R}^b_h(\boldsymbol{\Theta}_{hb})\mathbf{S}^{\mathrm{T}}(\mathbf{r}^h_{so_b})]\dot{\boldsymbol{\xi}}. \tag{67}$$

Zero speed:

$$\mathbf{v}^b_{no_b} = [\mathbf{R}^b_h(\boldsymbol{\Theta}_{hb}) \quad \mathbf{R}^b_h(\boldsymbol{\Theta}_{hb})\mathbf{S}^{\mathrm{T}}(\mathbf{r}^h_{so_b})]\dot{\boldsymbol{\xi}}. \tag{68}$$

8.2 Angular Velocity Transformation

The angular velocity transformation between the h- and the b-frame is similar to that bewteen the the n- and the b-frame, except that we need to use the perturbation yaw, pitch and yaw:

$$\boldsymbol{\omega}_{hb}^{b} = \mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta}_{hb})^{-1}\dot{\boldsymbol{\Theta}}_{hb}, \tag{69}$$

$$\boldsymbol{\omega}_{hb}^{b} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \qquad \dot{\boldsymbol{\Theta}}_{hb} = \begin{bmatrix} \dot{\xi}_4 \\ \dot{\xi}_5 \\ \dot{\xi}_6 \end{bmatrix}, \tag{70}$$

and

$$\mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta}_{hb}) = \begin{bmatrix} 1 & s\xi_4 t\xi_5 & c\xi_4 t\xi_5 \\ 0 & c\xi_4 & -s\xi_5 \\ 0 & s\xi_4/c\xi_5 & c\xi_4/c\xi_5 \end{bmatrix}, \tag{71}$$

$$\mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta}_{hb})^{-1} = \begin{bmatrix} 1 & 0 & -s\xi_5 \\ 0 & c\xi_4 & c\xi_5 s\xi_4 \\ 0 & -s\xi_4 & c\xi_4 c\xi_5 \end{bmatrix} \tag{72}$$

with $s \equiv \sin(\cdot)$, $c \equiv \cos(\cdot)$, $t \equiv \tan(\cdot)$ and $c\xi_5 \neq 0$.

9 The h to b Kinematic Transformation

Let us combine the results of the previous sections. Thus under Assumptions 1 and 2, we have

Transit:

$$\boldsymbol{\nu} = \mathbf{J}_h^b(\boldsymbol{\Theta}_{hb}, \mathbf{r}_{so_b}^h)\,\dot{\boldsymbol{\xi}} + \begin{bmatrix} \mathbf{R}_n^b(\boldsymbol{\Theta}_{nb}) \\ \mathbf{0}_{3\times 3}. \end{bmatrix} \bar{\mathbf{v}}_{no_h}^n \tag{73}$$

Zero speed:

$$\boldsymbol{\nu} = \mathbf{J}_h^b(\boldsymbol{\Theta}_{hb}, \mathbf{r}_{so_b}^h)\,\dot{\boldsymbol{\xi}} \tag{74}$$

with

$$\mathbf{J}_h^b(\boldsymbol{\Theta}_{hb}, \mathbf{r}_{so_b}^h) \triangleq \begin{bmatrix} \mathbf{R}_h^b(\boldsymbol{\Theta}_{hb}) & \mathbf{R}_h^b(\boldsymbol{\Theta}_{hb})\mathbf{S}^{\mathrm{T}}(\mathbf{r}_{so_b}^h) \\ \mathbf{0}_{3\times 3} & \mathbf{T}_{\boldsymbol{\Theta}}(\boldsymbol{\Theta}_{hb})^{-1} \end{bmatrix} \tag{75}$$

10 The b to h Kinematic Transformation

Here we will obtain the inverse of (75), such that

$$\dot{\boldsymbol{\xi}} = \mathbf{J}_b^h(\boldsymbol{\Theta}_{hb}, \mathbf{r}_{so_b}^h)\left(\boldsymbol{\nu} - \begin{bmatrix} \mathbf{R}_n^b(\boldsymbol{\Theta}_{nb}) \\ \mathbf{0}_{3\times 3}. \end{bmatrix} \bar{\mathbf{v}}_{no_h}^n\right) \tag{76}$$

To obtain this inverse, we need the following result: for a non-singular partitioned matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \tag{77}$$

the inverse can be expressed as (Horn and Johnson, 1985):

$$\mathbf{A}^{-1} = \begin{bmatrix} [\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}]^{-1} & \mathbf{A}_{11}^{-1}\mathbf{A}_{12}[\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} - \mathbf{A}_{22}]^{-1} \\ [\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} - \mathbf{A}_{22}]^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & [\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}]^{-1} \end{bmatrix} \tag{78}$$

For the particular case in which $\mathbf{A}_{21} = \mathbf{0}$, the above reduces to

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0} & \mathbf{A}_{22}^{-1} \end{bmatrix} \tag{79}$$

Direct application to this result to (75) yields

$$\mathbf{J}_b^h(\mathbf{\Theta}_{hb}, \mathbf{r}_{so_b}^h) \triangleq \mathbf{J}_h^b(\mathbf{\Theta}_{hb}, \mathbf{r}_{so_b}^h)^{-1} = \begin{bmatrix} \mathbf{R}_b^h(\mathbf{\Theta}_{hb}) & \mathbf{S}(\mathbf{r}_{so_b}^h)\mathbf{T}_{\mathbf{\Theta}}(\mathbf{\Theta}_{hb}) \\ \mathbf{0}_{3\times3} & \mathbf{T}_{\mathbf{\Theta}}(\mathbf{\Theta}_{hb}) \end{bmatrix}. \tag{80}$$

Hence,

$$\dot{\boldsymbol{\xi}} = \mathbf{J}_b^h(\mathbf{\Theta}_{hb}, \mathbf{r}_{so_b}^h) \left(\boldsymbol{\nu} - \begin{bmatrix} \mathbf{R}_n^b(\mathbf{\Theta}_{nb}) \\ \mathbf{0}_{3\times3}. \end{bmatrix} \bar{\mathbf{v}}_{no_h}^n \right) \tag{81}$$

with

$$\mathbf{J}_b^h(\mathbf{\Theta}_{hb}, \mathbf{r}_{so_b}^h) = \begin{bmatrix} \mathbf{R}_b^h(\mathbf{\Theta}_{hb}) & \mathbf{S}(\mathbf{r}_{so_b}^h)\mathbf{T}_{\mathbf{\Theta}}(\mathbf{\Theta}_{hb}) \\ \mathbf{0}_{3\times3} & \mathbf{T}_{\mathbf{\Theta}}(\mathbf{\Theta}_{hb}) \end{bmatrix}. \tag{82}$$

11 Small Angle Approximations for the h to b Kinematic Transformation

The small angle approximation of the rotation matrix and the inverse of the angular velocity transformations from h to b are

$$\mathbf{R}_h^b(\mathbf{\Theta}_{hb}) \approx \begin{bmatrix} 1 & \xi_6 & -\xi_5 \\ -\xi_6 & 1 & \xi_4 \\ \xi_5 & -\xi_4 & 1 \end{bmatrix} \quad \mathbf{T}_{\mathbf{\Theta}}(\mathbf{\Theta}_{hb})^{-1} \approx \begin{bmatrix} 1 & 0 & -\xi_5 \\ 0 & 1 & \xi_4 \\ 0 & -\xi_4 & 1 \end{bmatrix}. \tag{83}$$

The vector $\mathbf{r}_{so_b}^h$ can is

$$\mathbf{r}_{so_b}^h = \mathbf{R}_s^h(\mathbf{\Theta}_{hb})\mathbf{r}_{so_b}^s \approx \begin{bmatrix} 1 & -\xi_6 & \xi_5 \\ \xi_6 & 1 & -\xi_4 \\ -\xi_5 & \xi_4 & 1 \end{bmatrix} \begin{bmatrix} x_{so_b}^s \\ y_{so_b}^s \\ z_{so_b}^s \end{bmatrix}. \tag{84}$$

Thus,

$$\mathbf{r}_{so_b}^h = \begin{bmatrix} x_{so_b}^h \\ y_{so_b}^h \\ z_{so_b}^h \end{bmatrix} \approx \begin{bmatrix} (x_{so_b}^s - \xi_6 y_{so_b}^s + \xi_5 z_{so_b}^s) \\ (\xi_6 x_{so_b}^s + y_{so_b}^s - \xi_4 z_{so_b}^s) \\ (-\xi_5 x_{so_b}^s + \xi_4 y_{so_b}^s + z_{so_b}^s) \end{bmatrix}. \tag{85}$$

With

$$\mathbf{S}(\mathbf{r}^h_{so_b})^T = \begin{bmatrix} 0 & z^h_{so_b} & -y^h_{so_b} \\ -z^h_{so_b} & 0 & x^h_{so_b} \\ y^h_{so_b} & -x^h_{so_b} & 0 \end{bmatrix}. \tag{86}$$

After multiplying and by considering only the linear terms we find that for the case of **zero forward speed**

$$\boldsymbol{\nu} \approx \begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{S}^{\mathrm{T}}(\mathbf{r}^s_{so_b}) \\ \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} \end{bmatrix} \dot{\boldsymbol{\xi}}, \tag{87}$$

Componentwise,

$$\begin{aligned} u &\approx \dot{\xi}_1 + z^s_{so_b}\dot{\xi}_5 - y^s_{so_b}\dot{\xi}_6 \\ v &\approx \dot{\xi}_2 - z^s_{so_b}\dot{\xi}_4 + x^s_{so_b}\dot{\xi}_6 \\ w &\approx \dot{\xi}_3 + y^s_{so_b}\dot{\xi}_4 - x^s_{so_b}\dot{\xi}_5 \\ p &\approx \dot{\xi}_4 \\ q &\approx \dot{\xi}_5 \\ r &\approx \dot{\xi}_6 \end{aligned} \tag{88}$$

The accelerations then become

$$\begin{aligned} \dot{u} &\approx \ddot{\xi}_1 + z^s_{so_b}\ddot{\xi}_5 - y^s_{so_b}\ddot{\xi}_6 \\ \dot{v} &\approx \ddot{\xi}_2 - z^s_{so_b}\ddot{\xi}_4 + x^s_{so_b}\ddot{\xi}_6 \\ w &\approx \ddot{\xi}_3 + y^s_{so_b}\ddot{\xi}_4 - x^s_{so_b}\ddot{\xi}_5 \\ \dot{p} &\approx \ddot{\xi}_4 \\ \dot{q} &\approx \ddot{\xi}_5 \\ \dot{r} &\approx \ddot{\xi}_6 \end{aligned} \tag{89}$$

Let us now consider the case of forward speed. To do this we need to consider the last term in (73). Here we cannot make the assumption of small angle only for roll and pitch, but not for yaw. Thus, we will consider

$$\begin{aligned} \psi &= \overline{\psi} + \delta\psi \\ \phi &= \overline{\phi} + \delta\phi \\ \theta &= \overline{\theta} + \delta\theta, \end{aligned} \tag{90}$$

in which $\overline{\phi} = \overline{\theta} = 0$.

We can consider now a series expansion of the rotation matrix:

$$\mathbf{R}_n^b(\boldsymbol{\Theta}_{nb}) \approx \mathbf{R}_n^b(\overline{\boldsymbol{\Theta}}_{nb}) + \left.\frac{d\mathbf{R}_n^b(\boldsymbol{\Theta}_{nb})}{d\boldsymbol{\Theta}_{nb}}\right|_{\boldsymbol{\Theta}_{nb}=\overline{\boldsymbol{\Theta}}_{nb}} \delta\boldsymbol{\Theta}_{nb}. \tag{91}$$

The first term becomes

$$\mathbf{R}_n^b(\overline{\boldsymbol{\Theta}}_{nb}) = \begin{bmatrix} \cos\overline{\psi} & -\sin\overline{\psi} & 0 \\ \sin\overline{\psi} & \cos\overline{\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{92}$$

The second term becomes

$$\left.\frac{d\mathbf{R}_n^b(\boldsymbol{\Theta}_{nb})}{d\boldsymbol{\Theta}_{nb}}\right|_{\boldsymbol{\Theta}_{nb}=\overline{\boldsymbol{\Theta}}_{nb}} \delta\boldsymbol{\Theta}_{nb} = \begin{bmatrix} -\sin(\overline{\psi})\delta\psi & -\cos(\overline{\psi})\delta\psi & 0 \\ \cos(\overline{\psi})\delta\psi & -\sin(\overline{\psi})\delta\psi & 0 \\ 0 & 0 & \delta\psi \end{bmatrix}. \tag{93}$$

Therefore, the second term in (73), which appears due to the average speed of the ship is

$$\begin{bmatrix} \cos\overline{\psi} & -\sin\overline{\psi} & 0 \\ \sin\overline{\psi} & \cos\overline{\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{\mathbf{v}}_{no_h}^n + \begin{bmatrix} -\sin(\overline{\psi})\delta\psi & -\cos(\overline{\psi})\delta\psi & 0 \\ \cos(\overline{\psi})\delta\psi & -\sin(\overline{\psi})\delta\psi & 0 \\ 0 & 0 & \delta\psi \end{bmatrix} \bar{\mathbf{v}}_{no_h}^n. \tag{94}$$

The first term of (94) gives a constant velocity in the body-fixed frame:

$$\overline{\boldsymbol{\nu}} \triangleq \begin{bmatrix} \cos\overline{\psi} & -\sin\overline{\psi} & 0 \\ \sin\overline{\psi} & \cos\overline{\psi} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \bar{\mathbf{v}}_{no_h}^n. \tag{95}$$

The final step is obtained by noting that due to the definition of the h-frame the following holds $\delta\psi = \xi_6$. From this, it follows that the small angle approximation of (73) is

$$\boxed{\boldsymbol{\nu} - \overline{\boldsymbol{\nu}} \approx \begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{S}^{\mathrm{T}}(\mathbf{r}_{so_b}^s) \\ \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} \end{bmatrix} \dot{\boldsymbol{\xi}} + \begin{bmatrix} -\sin(\overline{\psi})\xi_6 & -\cos(\overline{\psi})\xi_6 & 0 \\ \cos(\overline{\psi})\xi_6 & -\sin(\overline{\psi})\xi_6 & 0 \\ 0 & 0 & \xi_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \bar{\mathbf{v}}_{no_h}^n. \quad (96)}$$

If we consider

$$\bar{\mathbf{v}}_{no_h}^n \triangleq \begin{bmatrix} U \\ V \\ 0 \end{bmatrix}. \tag{97}$$

then componentwise

$$\boxed{\begin{aligned} u - \bar{u} &\approx \dot{\xi}_1 + z_{so_b}^s \dot{\xi}_5 - y_{so_b}^s \dot{\xi}_6 - \sin(\overline{\psi}) U \xi_6 - \cos(\overline{\psi}) V \xi_6 \\ v - \bar{v} &\approx \dot{\xi}_2 - z_{so_b}^s \dot{\xi}_4 + x_{so_b}^s \dot{\xi}_6 + \cos(\overline{\psi}) U \xi_6 - \sin(\overline{\psi}) V \xi_6 \\ w - \bar{w} &\approx \dot{\xi}_3 + y_{so_b}^s \dot{\xi}_4 - x_{so_b}^s \dot{\xi}_5 \\ p &\approx \dot{\xi}_4 \\ q &\approx \dot{\xi}_5 \\ r &\approx \dot{\xi}_6 \end{aligned} \qquad (98)}$$

The acceleration then become

$$\boxed{\begin{aligned} \dot{u} &\approx \ddot{\xi}_1 + z_{so_b}^s \ddot{\xi}_5 - y_{so_b}^s \ddot{\xi}_6 - \sin(\overline{\psi}) U \dot{\xi}_6 - \cos(\overline{\psi}) V \dot{\xi}_6 \\ \dot{v} &\approx \ddot{\xi}_2 - z_{so_b}^s \ddot{\xi}_4 + x_{so_b}^s \ddot{\xi}_6 + \cos(\overline{\psi}) U \dot{\xi}_6 - \sin(\overline{\psi}) V \dot{\xi}_6 \\ w &\approx \ddot{\xi}_3 + y_{so_b}^s \ddot{\xi}_4 - x_{so_b}^s \ddot{\xi}_5 \\ \dot{p} &\approx \ddot{\xi}_4 \\ \dot{q} &\approx \ddot{\xi}_5 \\ \dot{r} &\approx \ddot{\xi}_6 \end{aligned} \qquad (99)}$$

12 Application Example: Motion Superposition Model for DP

Let's take an example in which we use SHIPX-VERES (Fathi, 2004) to obtain the motion RAOs of a vessel, and we would like to incorporate wave-induced motion in the equations of motion for testing the design of a control system for dynamic positioning.

One way of achieving this, is to consider first the motion due to the control action:

$$\begin{aligned} \mathbf{M}^b \dot{\boldsymbol{\nu}}_{LF} + \mathbf{D}^b \boldsymbol{\nu}_{LF} + \mathbf{g}(\boldsymbol{\eta}_{LF}) &= \boldsymbol{\tau}_c^b + \boldsymbol{\tau}_{env}^b \\ \dot{\boldsymbol{\eta}}_{LF} &= \mathbf{J}_b^n(\boldsymbol{\Theta}_{nb}) \boldsymbol{\nu}_{LF}, \end{aligned} \tag{100}$$

and use motion superposition:

$$\boldsymbol{\eta} = \boldsymbol{\eta}_{LF} + \boldsymbol{\eta}_w. \tag{101}$$

In the above equations $\boldsymbol{\eta}$ is total motion of the vessel, $\boldsymbol{\eta}_{LF}$ is the low-frequency motion, and $\boldsymbol{\eta}_w$ the wave-induced motion. The generalised forces on the right hand side of (100) are the control forces and the slowly-varying environmental loads (current, wind wave drift induced forces). This model is based on the linearity assumption of the equations of motion.

The motion due to the waves can be obtained by integrating the following

$$\dot{\boldsymbol{\eta}}_w = \mathbf{J}_b^n(\boldsymbol{\Theta}_{nb})\mathbf{J}_h^b(\boldsymbol{\Theta}_{hb}, \mathbf{r}_{so_b}^h)\,\dot{\boldsymbol{\xi}}, \tag{102}$$

where $\dot{\boldsymbol{\xi}}$ is the motion of the vessel due to waves calculated in the h-frame.

There is one more step that needs to be taken. In Veres, the motion transfer functions are given with respect to a global frame, which has a different orientation than the h-frame used to derive the transformation above. The *Veres global* reference frame is defined as follows (Fathi, 2004):

The $x - y$ plane coincides with the still water plane, the $x - z$ plane coincides with the center plane of the vessel. The positive x-axis is directed towards stern, the y-axis is directed to starboard, and the positive z-axis upwards and through the centre of gravity.

Therefore, if we denote the Veres coordinates by $\boldsymbol{\xi}_v$, we have that

$$\dot{\boldsymbol{\xi}} = \mathbf{D}_v^h\,\dot{\boldsymbol{\xi}}_v, \tag{103}$$

where

$$\mathbf{D}_h^v = diag(-1, 1, -1, -1, 1, -1). \tag{104}$$

The time series of $\dot{\boldsymbol{\xi}}_v$ can be obtained as a sum of sinusoids with amplitudes and phases calculated from the RAOs and the particular wave spectrum—see (Perez, 2005).

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